QUANTUM DIFFERENTIAL OPERATORS ON $\mathbb{k}[x]$

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ABSTRACT. Following the definition given in [LR1], we compute the ring of quantum differential operators on the polynomial ring in 1 variable. We further study this ring.

0. INTRODUCTION

Quantum differential operators were defined by V.A.Lunts and A.L.Rosenberg in their work [LR1], as part of their project on Localization for Quantum groups ([LR2]). Quantum differential operators are defined on graded rings, graded by an abelian group. Let $\mathbb{k}$ be a field and R an associative $\mathbb{k}$-algebra, graded by an abelian group Γ. The ring of quantum differential operators (or q-differential operators), denoted by $D_q(R)$ is defined for a fixed bicharacter $\beta : \Gamma \times \Gamma \to \mathbb{k}^*$. For each $a \in \Gamma$, there is an automorphism σ_a of R given by $\sigma_a(r_b) = \beta(a,b)r_b$, where degree of $r_b = b$. Turns out that each σ_a and the corresponding left-σ_a-derivation is a quantum differential operator. (This is shown in section 1).

In this paper we compute the ring of q-differential operators on a polynomial ring in 1 variable. We also show a relationship between this ring and the quantum group on sl_2.

In the first section we cover the preliminaries required in the rest of the paper

In the second section, we consider the polynomial ring on one variable, graded by $\mathbb{Z}$. Let q be transcendental over $\mathbb{Q}$ and $\mathbb{k}$ be a field containing $\mathbb{Q}(q)$. We let $R = \mathbb{k}[x]$, graded by the group of integers as $deg(x) = 1$. The bicharacter β is defined as $\beta(n,m) = q^{nm}$. Using this set up, we show in Theorem 2.0.1 that $D_q(R)$ is generated over $\mathbb{k}$by $x, \partial, \partial^\beta, \partial^{\beta^{-1}}$, where ∂ is the usual derivation,

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$(\partial(x^n) = nx^{n-1})$, ∂^β is the left σ_1-derivation $(\partial^\beta(x^n) = (1+q+\cdots+q^{n-1})x^{n-1})$ and $\partial^{\beta^{-1}}$ is the left σ_{-1}-derivation $(\partial^\beta(x^n) = (1+\frac{1}{q}+\cdots+\frac{1}{q^{n-1}})x^{n-1})$.

Here, we also show that $D_q(R)$ is a simple ring (2.0.2), a result similar to that of usual differential operators on R (characteristic 0).

In section 3, our main goal is to write $D_q(R)$ in terms of generators and relations, given in theorem 3.0.3; which is,

Theorem. *The ring D_q of q-differential operators on R is the $\Bbbk$-algebra generated by x, $\partial^{\beta^{-1}}$, $\partial^{\beta^0}(=\partial)$, and ∂^{β^1} subject to the relations*

$$\partial^{\beta^a}x - q^a x\partial^{\beta^a} = 1, \quad \partial^{\beta^a}x\partial^{\beta^b} = \partial^{\beta^b}x\partial^{\beta^a}, \quad \textit{and} \quad \partial^{\beta^{-1}} - q\partial^\beta = (1-q)\partial^{\beta^{-1}}x\partial^\beta.$$

Other results of independent interest are also proved in this section. If $\tau = x\partial$ and $\sigma = \sigma_1$, and $(D_q)_0$ denotes the q-differential operators of degree 0, then lemma 3.0.5 shows that $(D_q)_0$ is localization of the polynomial ring $\Bbbk[\tau,\sigma]$ at the element σ; that is $(D_q)_0 = \Bbbk[\tau,\sigma,\sigma^{-1}]$. A corollary (3.0.2) to this lemma is the fact that $D_q(R)$ is a domain (again similar to the result in usual differential operators).

In section 4, we generalize the computation of $D_q(R)$ to the case when R is a polynomial ring in n variables, graded by $\mathbb{Z}^n$, and the bicharacter is product of n-bicharacters of the above type. That is, we let $\Bbbk$ let be a field containing $\mathbb{Q}$ and n transcendental elements $q_1, q_2, \cdots, q_n$, and define

$$\beta((a_1,a_2,\cdots,a_n),(b_1,b_2,\cdots,b_n)) = q_1^{a_1b_1}q_2^{a_2b_2}\cdots q_n^{a_nb_n}.$$

Here, again as in the case of usual differential operators, one can define for each $i, 1 \le i \le n$, the operators $\partial_i^{\beta^{-1}}, \partial_i, \partial_i^{\beta^1}$. We show in theorem 4.0.4 that these operators along with R generate $D_q(R)$.

In the last section we explain relationship between quantum group on $sl_2(\Bbbk)$, denoted by $U_q(sl_2)$, and the global q-differential operators on $\mathbb{P}^1$. Here, we fix $\Gamma = \mathbb{Z}$ and $\beta(n,m) = q^{nm}$. The q-differential operators $D_q(\Bbbk[x])$ and $D_q(\Bbbk[x^{-1}])$ extend to q-differential operators on $\Bbbk[x,x^{-1}]$ by theorem 3.2.2. of [LR1]. So, we

let $\Gamma_q(\mathbb{P}^1)$ to be the kernel of

$$D_q(\Bbbk[x]) \bigoplus D_q(\Bbbk[x^{-1}]) \to D_q(\Bbbk[x, x^{-1}])$$
$$(\varphi_1, \varphi_2) \mapsto \varphi_1 - \varphi_2.$$

There is a homomorphism $\eta : U_q(sl_2) \to \Gamma_q(\mathbb{P}^1)$, which generalizes the homomorphism of the enveloping algebra of sl_2 to the ring of global sections of usual differential operators on $\mathbb{P}^1$. But unlike the enveloping algebra case (characteristic 0), the homomorphism η is not a surjection. So, next we replace $\Bbbk$by the ring $\mathcal{A} = \mathbb{Q}[q, q^{-1}]_{(q-1)}$ a local ring where $q-1$ is not invertible. We consider the corresponding rings $U_q(sl_2)$ and Γ_q with $\mathcal{A}$ as the base ring and consider their respective inverse limits with respect to the ideal $(q-1)$. The homomorphism η gives rise to a homomorphism of inverse limits, denoted by $\hat{\eta}$. We show that $\hat{\eta}$ is a surjection in theorem 5.4.1. We would like to point out that some of the formulae derived here have already been derived in the literature of Mathematical Physics, see for example [GP].

We thank Professor Valery Lunts for suggesting this problem especially for his ideas regarding the relationship with quantum group on sl_2.

1. Preliminaries

Throughout this paper, let q be transcendental over $\mathbb{Q}$ and $\Bbbk$ be a field containing $\mathbb{Q}(q)$. Let R be a $\Bbbk$-algebra which is graded by an abelian group Γ. That is, $R = \oplus_{\mathbf{a}\in\Gamma} R_{\mathbf{a}}$ and $R_{\mathbf{a}} R_{\mathbf{b}} \subseteq R_{\mathbf{a}+\mathbf{b}}$.

A $\Bbbk$-linear endomorphism φ of R is called *homogeneous of degree* $\mathbf{a} \in \Gamma$ if and only if for every $\mathbf{b} \in \Gamma$, $\varphi(R_{\mathbf{b}}) \subseteq R_{\mathbf{a}+\mathbf{b}}$. We will say φ is a *graded endomorphism* if and only if it is the sum of homogeneous endomorphisms. The $\Bbbk$-vector space of all graded endomorphisms of R will be denoted $\mathrm{grHom}_{\Bbbk}(R, R)$.

For each $r \in R$, we have two endomorphisms of R, left multiplication by r and right multiplication by r. We will denote these by λ_r and ρ_r respectively. When r is homogeneous of degree $\mathbf{a}$, so are λ_r and ρ_r. Hence, for any r, the maps λ_r and ρ_r are graded endomorphisms. The homomorphism $r \mapsto \lambda_r$ embeds

R into $\mathrm{grHom}_{\Bbbk}(R,R)$. This enables us to define right and left actions of R on $\mathrm{grHom}_{\Bbbk}(R,R)$ by $r \cdot \varphi \cdot s = \lambda_r \varphi \lambda_s$. Since a right R action is equivalent to a left action of the oposite ring R^{o}, we have a left action of the enveloping algebra $R^{\mathrm{e}} = R \bigotimes_{\Bbbk} R^{\mathrm{o}}$ on $\mathrm{grHom}_{\Bbbk}(R,R)$.

A bicharacter on Γ is a function β from $\Gamma \times \Gamma$ to $\Bbbk^{\times}$ such that, in each argument, β is a group homomorphism. To each bicharacter we can associate a family of automorphisms of $\mathrm{grHom}_{\Bbbk}(R,R)$ as follows. For each $\mathbf{a} \in \Gamma$, define $\widetilde{\sigma}_{\mathbf{a}}$ so that for any homogeneous φ of degree $\mathbf{b}$, $\widetilde{\sigma}_{\mathbf{a}}(\varphi) = \beta(\mathbf{a},\mathbf{b})\varphi$. This extends linearly to all of $\mathrm{grHom}_{\Bbbk}(R,R)$.

Now with each $\widetilde{\sigma}_{\mathbf{a}}$ we can define a pairing in $\mathrm{grHom}_{\Bbbk}(R,R)$. Let φ and ψ be graded endomorphisms. Define the $\mathbf{a}$-*twisted bracket* to be

$$[\varphi,\psi]_{\mathbf{a}} = \varphi\psi - \widetilde{\sigma}_{\mathbf{a}}(\psi)\varphi.$$

The reader is cationed that this will be a Lie bracket only if $\mathbf{a}$ is the identity $\mathbf{0}$. In this case, the $\mathbf{0}$-twisted bracket will simply be denoted by $[\cdot,\cdot]$.

In a similar way, we can define a family of endomorphisms of R. If $r \in R$ is homogeneous of degree $\mathbf{b}$, then define $\sigma_{\mathbf{a}}(r) = \beta(\mathbf{a},\mathbf{b})r$. This can be extended linearly to all of R making $\sigma_{\mathbf{a}}$ an homogeneous endomorphism of degree 0. Then for any $r \in R$, $\widetilde{\sigma}_{\mathbf{a}}(\lambda_r) = \lambda_{\sigma_{\mathbf{a}}(r)}$. More generally, for any graded endomorphism φ we have

$$\widetilde{\sigma}_{\mathbf{a}}(\varphi)\sigma_{\mathbf{a}} = \sigma_{\mathbf{a}}\varphi. \tag{1.0.1}$$

1.1. **Quantum differential operators.** Following [LR1], we define $D_q^0(R)$, the *q-centre* of $\mathrm{grHom}_{\Bbbk}(R,R)$, to be the smallest R^{e} submodule of $\mathrm{grHom}_{\Bbbk}(R,R)$ containing the $\Bbbk$-space

$$\begin{aligned} \mathrm{Z}_q^0 = \Bbbk\langle \text{ homogeneous } \varphi \mid & \text{ there is some } \mathbf{a} \in \Gamma \text{ such that for} \\ & \text{any } r \in R,\ [\varphi,\lambda_r]_{\mathbf{a}} = 0\rangle. \end{aligned}$$

Suppose that we have defined already the R^{e}-submodule $D_q^n(R)$ of $\mathrm{grHom}_{\Bbbk}(R,R)$. Then the R^{e}-module $D_q^{n+1}(R)$ is defined to be the smallest R^{e}-submodule containing the $\Bbbk$-space

$$\mathrm{Z}_q^{n+1} = \Bbbk\langle \text{ homogeneous } \varphi \mid \text{there is some } \mathbf{a} \in \Gamma \text{ such that for any}$$
$$r \in R,\ [\varphi, \lambda_r]_{\mathbf{a}} \in D_q^n(R)\rangle.$$

We say φ is a *q-differential operator of order n* if and only if $\varphi \in D_q^n(R)$. Finally, the set of all q-differential operators is defined to be $D_q(R) = \bigcup D_q^n(R)$.

1.2. **The q-centre of $D_q(R)$.** As is the case with differential operators over a commutative ring, the composite of a q-differential operator of order n with one of order m is a q-differential operator of order $m+n$ [LR1, 3.1.8]. This shows that $D_q^0(R)$ is a ring and each $D_q^n(R)$ is a $D_q^0(R)$-module.

The structure of $D_q^0(R)$ is apparent:

Lemma 1.2.1. *The ring $D_q^0(R)$ is a Γ-graded $\Bbbk$-algebra generated by*

$$\{\lambda_r\rho_s\sigma_{\mathbf{a}} \mid\ \mathbf{a} \in \Gamma \textit{ and } r,s \in R\ \}.$$

Proof. First, we will show that Z_q^0 is generated as a $\Bbbk$-module by

$$\{\rho_r\sigma_{\mathbf{a}} \mid\ r \text{ is homogeneous }\}.$$

Let φ be a homogeneous endomorphism in the q-centre of $\mathrm{grHom}_{\Bbbk}(R,R)$. Then, $[\varphi,\lambda_s]_{\mathbf{a}} = 0$ for some $\mathbf{a} \in \Gamma$ and all s in R. Since φ is homogeneous, $\varphi(1)$ is some homogeneous r of the same degree as φ. Then $\varphi(s) = \varphi\lambda_s(1) = \lambda_{\sigma_{\mathbf{a}}(s)}\varphi(1) = \sigma_{\mathbf{a}}(s)r$. Since φ has the same value on s as $\rho_r\sigma_{\mathbf{a}}$, we have $\varphi = \rho_r\sigma_{\mathbf{a}}$.

To prove Z_q^0 contains all of the prescribed operators, let r be any homogeneous element and $\mathbf{a}$ any element of Γ. Since left multiplication and right multiplication commute, for any s we have $[\rho_r\sigma_{\mathbf{a}},\lambda_s]_{\mathbf{a}} = \rho_r[\sigma_{\mathbf{a}},\lambda_s]_{\mathbf{a}}$. Furthermore, by (1.0.1) we have $[\sigma_{\mathbf{a}},\lambda_s]_{\mathbf{a}} = 0$. Hence, $\rho_r\sigma_{\mathbf{a}}$ is in the q-centre of $\mathrm{grHom}_{\Bbbk}(R,R)$.

Now, by its definition $D_q^0(R)$ is the $\Bbbk$-subspace of $\mathrm{grHom}_{\Bbbk}(R,R)$ spanned by $\{\lambda_r\rho_s\sigma_{\mathbf{a}}\lambda_t\}$. However, λ_t commutes with $\sigma_{\mathbf{a}}$ up to multiplication by a scalar, and

commutes with ρ_s without incident, so $D_q^0(R)$ can be generated by the given operators as required. $\square$

Corollary 1.2.1. *The module $D_q^{n+1}(R)$ of q-differential operators of order $n+1$ are generated over $D_q^0(R)$ by*

$$\mathrm{Z'}_q^{n+1} = \{ \text{ homogeneous } \varphi \mid \text{ such that for any } r \in R, [\varphi, \lambda_r] \in D_q^n(R)\}.$$

Proof. It is clear that $\mathrm{Z'}_q^{n+1}$ is contained in Z_q^{n+1} so we only need to show that every $\varphi \in \mathrm{Z}_q^{n+1}$ is in the $D_q^0(R)$ span of $\mathrm{Z'}_q^{n+1}$. To this end, suppose for some $a \in \Gamma$ and all $r \in R$ that $[\varphi, \lambda_r]_{\mathbf{a}} \in D_q^n(R)$. Then we have for any $b \in \Gamma$,

$$\begin{aligned}
[\varphi\sigma_{\mathbf{b}}, \lambda_r]_{\mathbf{a}+\mathbf{b}} =& \varphi\sigma_{\mathbf{b}}\lambda_r - \widetilde{\sigma}_{\mathbf{a}+\mathbf{b}}(\lambda_r)\varphi\sigma_{\mathbf{b}} \\
=& \varphi\widetilde{\sigma}_{\mathbf{b}}(\lambda_r)\sigma_{\mathbf{b}} - \widetilde{\sigma}_{\mathbf{a}}(\widetilde{\sigma}_{\mathbf{b}}(\lambda_r))\varphi\sigma_{\mathbf{b}} \\
=& [\varphi, \widetilde{\sigma}_{\mathbf{b}}(\lambda_r)]_{\mathbf{a}}\sigma_{\mathbf{b}}.
\end{aligned}$$

Since $\sigma_{\mathbf{b}}$ is an automorphism of R, $[\varphi, \widetilde{\sigma}_{\mathbf{b}}(\lambda_r)]_{\mathbf{a}} \in D_q^n(R)$. Hence $[\varphi\sigma_{\mathbf{b}}, \lambda_r]_{\mathbf{a}+\mathbf{b}} \in D_q^n(R)$. Since Γ is a group, we may put $\mathbf{b} = -\mathbf{a}$. Then $[\varphi\sigma_{-\mathbf{a}}, \lambda_r] \in D_q^n(R)$ for any $r \in R$. Hence $\varphi\sigma_{-\mathbf{a}} \in \mathrm{Z'}_q^{n+1}$. $\square$

Remark 1.2.1. *Typically, a bicharacter β is used to define an associative algebra structure on the tensor product of two graded algebras. Let $A = \oplus_{\mathbf{a}\in\Gamma} A_{\mathbf{a}}$ be a Γ-graded $\Bbbk$-algebra, and define A_Γ to be the algebra whose underlying set of elements is $A \bigotimes \Bbbk[\Gamma]$, and whose multiplicative structure is given as follows: If we define the grading automorphisms $\sigma_{\mathbf{a}}$ as above, then define*

$$r \otimes \mathbf{a} \cdot s \otimes \mathbf{b} = r(\sigma_{\mathbf{a}}(s)) \otimes (\mathbf{a} + \mathbf{b}).$$

Requiring β to be a group homomorphism in each variable ensures that A_Γ is associative. We call this the crossed-product algebra determined by β.

It is not surprising in light of this that $D_q^0(R)$ has the structure of a crossed-product algebra. Indeed, if we put $A = R \bigotimes_{Z(R)} R^{\mathrm{o}}$ where $Z(R)$ is the center of R, then $D_q^0(R) = A_\Gamma$.

1.3. **Left β-derivations.** The definition of a derivation of a ring is quite standard and can be applied to commutative and noncommutative rings alike. A derivation is always a differential operator, and, in the best of circumstances, the derivations generate all other differential operators. However, since the q-differential operators form a broader class of graded endomorphisms, even in the best of circumstances, we will require more than just the derivations to generate all of the q-differential operators.

Definition. *A left β-derivation is a graded endomorphism φ of R such that φ obeys a "twisted" Leibniz's Rule. That is, for some $a \in \Gamma$ and any $r, s \in R$,*

$$\varphi(rs) = \varphi(r)s + \sigma_{\mathbf{a}}(r)\varphi(s).$$

Equivalently, we could define a left β-derivation to be an endomorphism φ such that for some $\mathbf{a} \in \Gamma$ and any $r \in R$ we have $[\varphi, \lambda_r]_{\mathbf{a}} = \lambda_{\varphi(r)}$. From this description, it becomes clear why let β-derivations are q-differential operators, and why the analogously defined *right* β-derivations, those φ for which there is an $\mathbf{a} \in \Gamma$ such that for all $r \in R$ we have $[\varphi, \rho_r]_{\mathbf{a}} = \rho_{\varphi(r)}$, bear little importance to us.

2. The q-Differential Operators on a Polynomial Ring in 1 Variable

Let $R = \Bbbk[x]$, and, to simplify notation, let $D_q^n = D_q^n(R)$ and $D_q = D_q(R)$. The ring R has a $\mathbb{Z}$-grading given by $\deg(x^a) = a$. Denote by R_a the a-th graded part $\Bbbk \cdot x^a$. Let $\beta : \mathbb{Z} \times \mathbb{Z} \to \Bbbk^\times$ be a bicharacter. Since $\beta(n,m) = (\beta(1,1))^{nm}$, we let $\beta(n,m) = q^{nm}$.

Since R is commutative, $R^{\mathrm{o}} = R$, and $D_q^0 = \Bbbk\langle r\sigma_a \mid \; r \in R \text{ and } a \in \mathbb{Z} \rangle$. In this section, we shall give an explicit description of D_q.

We begin with two lemmas which will help us recognize q-differential operators of order n.

Lemma 2.0.1. *If $[\varphi, x] \in D_q^n$ then $\varphi \in \mathrm{Z}_q'^{\,n+1}$.*

Proof. Since $[\cdot,\cdot]$ is a bracket, $[\varphi, \cdot]$ is a derivation. Hence, for any $m > 0$, we have

$$[\varphi, x^m] = [\varphi, x]x^{m-1} + x[\varphi, x^{m-1}].$$

Inductively, we have that if $[\varphi, x] \in D_q^n$ then $[\varphi, x^{m-1}] \in D_q^n$. Hence $[\varphi, x^m]$ is in the R-span of D_q^n, and so it is in D_q^n. It follows that for any $r \in R$, $[\varphi, r] \in D_q^n$. Hence $\varphi \in \mathrm{Z}_q^{\prime n+1}$. □

This means we do not have to test $[\varphi, \cdot]$ against all elements of R in order to determine whether or not φ is in $\mathrm{Z}_q^{\prime n+1}$. For general rings R, we have a similar result which we will not prove: $\varphi \in \mathrm{Z}_q^{\prime n+1}$ if and only if $[\varphi, \lambda_r] \in D_q^n$ for all generators r of R over $\Bbbk$.

Lemma 2.0.2. *Given $\varphi \in D_q(R)$, there exists integers $n, a_1, a_2, \cdots, a_n$ such that $n > 0$ and $[\cdots[[\varphi, x]_{a_1}, x]_{a_2}, \cdots, x]_{a_n} = 0$.*

Proof. Indeed, if $\varphi \in D_q^k(R)$, then

$$\varphi = \varphi_1 + \varphi_2 + \cdots + \varphi_l$$

for homogeneous $\varphi_i \in \mathrm{Z}_q^k$. Corresponding to each φ_i, there exists $b_i \in \mathbb{Z}$ such that $[\varphi_i, x]_{b_i} \in D_q^{k-1}(R)$. Hence,

$$[\cdots[[\varphi, x]_{b_1}, x]_{b_2}, \cdots, x]_{b_l} \in D_q^{k-1}(R).$$

Now induction completes the lemma. □

Let us now describe the left β-derivations on R.

Definition. *For each $a \in \mathbb{Z}$, define $\partial^{\beta^a} \in \mathrm{grHom}_{\Bbbk}(R, R)$ as*

$$\partial^{\beta^a}(x^b) = (1 + q^a + q^{2a} + \cdots + q^{a(b-1)})x^{b-1}.$$

We denote ∂^{β^1} by just ∂^β.

Simply by comparing values on x^m, one is led to the following:

Note.

1. For any positive integer a,

$$\partial^{\beta^a} = \left(\frac{1-q}{1-q^a}\right)\partial^\beta[1 + \sigma_1 + \cdots + \sigma_{a-1}].$$

2. When $a = 0$, ∂^{β^a} is ∂, the usual derivative defined by $\partial(x^b) = bx^{b-1}$.

3. For any positive integer a,

$$\partial^{\beta^{-a}} = \left(\frac{1-q}{1-q^{-a}}\right)\partial^{\beta^{-1}}[1+\sigma_{-1}+\cdots+\sigma_{1-a}].$$

4. For any integer a,

$$\partial^{\beta^{a}} = \sigma_a \partial^{\beta^{-a}}. \tag{2.0.1}$$

Lemma 2.0.3. *For any $a \in \mathbb{Z}$, the operator $\partial^{\beta^{a}}$ is in D_q^1.*

Proof. Let $\eta = [\partial^{\beta^{a}}, x]$. By Lemma 2.0.1, it is enough to show $\eta \in D_q^0$. For any $m \geq 0$,

$$\begin{aligned}\eta(x^m) =& \partial^{\beta^{a}}(x^{m+1}) - x\partial^{\beta^{a}}(x^m)\\ =&(1+q^a+\cdots+q^{am})x^m - x(1+q^a+\cdots+q^{a(m-1)})x^{m-1}\\ =&q^{am}x^m.\end{aligned}$$

Thus $\eta(r) = \sigma_a(r)$ for any $r \in R$. Hence $\eta = \sigma_a \in D_q^0$, as required. □

We will make use of the following notation. For each positive integer n and each multi-index $I = (a_1, a_2, \cdots, a_n)$, set

$$\partial^{\beta^{I}} = \partial^{\beta^{a_1}}\partial^{\beta^{a_2}}\cdots\partial^{\beta^{a_n}}. \tag{2.0.2}$$

Here we say $|I| = n$.

Corollary 2.0.1. *If $P = \partial^{\beta^{I}}$ and $n \geq |I|$, then $P \in D_q^n$.*

Proof. This follows from the fact that $P \in (D_q^1)^n \subset D_q^n$. □

We will prove the following theorem:

Theorem 2.0.1. *The ring D_q of q-differential operators on R is generated as a $\Bbbk$-algebra by R and the set $\{\partial^{\beta^{-1}}, \partial, \partial^{\beta}\}$.*

The proof is somewhat involved, so we have consigned the more technical parts of this theorem to a Lemma. We will use the following identity which can be easily checked by expanding the a-twisted bracket $[\cdot,\cdot]_a$ according to its definition:

$$[[\varphi,\psi]_a, x] = [[\varphi,x],\psi]_a + \varphi[\psi,x] - [\widetilde{\sigma}_a(\psi),x]\varphi. \tag{2.0.3}$$

Lemma 2.0.4. *If P is a q-differential operator comprised of monomials in $\{\partial^{\beta^a} \mid a \in \mathbb{Z}\}$ and b is any integer then there is a q-differential operator Q, also comprised of monomials in the ∂^{β^a}'s with coefficients in $\Bbbk$, such that $[Q,x] = P\sigma_b$. Thus, for $f \in R$, we have $[fQ,x] = fP\sigma_b$.*

Proof. It is enough to prove the lemma when P is a monomial, for if the lemma holds for monomials m_i, $i = 1,\dots,k$, and $P = m_1 + m_2 + \cdots + m_k$, then we can find differential operators M_i of the prescribed form such that $[M_i,x] = m_i\sigma_b$. Putting $Q = M_1 + M_2 + \cdots + M_k$ yields the desired result.

Now, suppose $P = d_n d_{n-1}\cdots d_2 d_1$ where each $d_i = \partial^{\beta^{a_i}}$. We will show by induction on n that there is a homogeneous q-differential operator Q of degree $n+1$, comprised of monomials in the ∂^{β^a}'s such that $[Q,x] = P\sigma_b$.

When $n = 0$ we have only one possibility: $P = 1$. Put $Q = \partial^{\beta^b}$. Since $[Q,x] = \sigma_b = P\sigma_b$, this proves the base case.

Before proceeding, we must address one technical point. We shall assume that if $b = -\sum_{i=1}^n a_i$ then every $a_i = 0$. To see that no generality is lost, suppose that $b = -\sum a_i$ and $a_n \neq 0$. Put $P' = \partial^{\beta^{-a_n}} d_{n-1}\cdots d_2 d_1$. By (2.0.1) $\partial^{\beta^{a_n}} = \sigma_{a_n}\partial^{\beta^{-a_n}}$. Thus we have

$$P\sigma_b = \sigma_{a_n}\partial^{\beta^{-a_n}} d_{n-1}\cdots d_1\sigma_b = cP'\sigma_{a_n+b}$$

for some nonzero constant c. If $[Q',x] = P'\sigma_{a_n+b}$ then $[cQ',x] = P\sigma_b$. However, since $a_n > 0$, we have $-(-a_n + a_{n-1} + \cdots + a_1) = 2a_n - \sum a_i \neq a_n + b$.

Now suppose the statement holds for monomials of degree $n-1$. Let

$$\begin{aligned} t_1 =& d_n d_{n-1} \cdots d_2, \\ t_2 =& d_1\ d_n\ \cdots\ d_3, \\ &\vdots \\ t_n =& d_{n-1} d_{n-2} \cdots d_1. \end{aligned}$$

That is $t_i d_i$ is the monomial obtained from P by a cyclic permutation of its factors. Then $P = t_1 d_1 = d_n t_n$, and $t_i d_i = d_{i-1} t_{i-1}$ when $i > 1$.

Let k_1, k_2, …, k_n be arbitrary integers. Then the series

$$[t_1, d_1]_{k_1} + q^{-k_1}[t_2, d_2]_{k_2} + q^{-k_1-k_2}[t_3, d_3]_{k_3} + \cdots + q^{-k_1-\cdots-k_{n-1}}[t_n, d_n]_{k_n}$$

is a telescoping series which reduces to

$$t_1 d_1 - q^{-\sum k_i} d_n t_n = (1 - q^{-\sum k_i})P.$$

By the induction hypothesis, there are T_i homogeneous in the ∂^{β^a}'s of degree n such that $[T_i, x] = t_i \sigma_b$. For each $i = 1, \ldots, n$, put $k_i = -na_i$. The homogeneity of the T_i ensures that for each i,

$$T_i[\partial^{\beta^{a_i}}, x] = T_i \sigma_{a_i} = q^{na_i} \sigma_{a_i} T_i = [\widetilde{\sigma}_{-na_i}(\partial^{\beta^{a_i}}), x] T_i,$$

Thus we get

$$T_i[d_i, x] - [\widetilde{\sigma}_{k_i}(d_i), x] T_i = 0.$$

We can use this and (2.0.3) to get

$$[[T_i, d_i]_{k_i}, x] = [t_i \sigma_b, d_i]_{k_i}$$

Thus, applying $[\cdot, x]$ to

$$\tilde{Q} = [T_1, d_1]_{k_1} + q^{-k_1}[T_2, d_2]_{k_2} + q^{-k_1-k_2}[T_3, d_3]_{k_3} + \cdots + q^{-(k_1+\cdots+k_{n-1})}[T_n, d_n]_{k_n}$$

will yield

$$[t_1\sigma_b, d_1]_{k_1} + q^{-k_1}[t_2\sigma_b, d_2]_{k_2} + q^{-(k_1+k_2)}[t_3\sigma_b, d_3]_{k_3} + \cdots + q^{-(k_1+\cdots+k_{n-1})}[t_n\sigma_b, d_n]_{k_n}. \tag{2.0.4}$$

We use the identity $[t_i\sigma_b, d_i]_{k_i} = q^{-b}[t_i, d_i]_{k_i-b}\sigma_b$ on (2.0.4) to get

$$q^{-b}\Big([t_1, d_1]_{k_1-b} + q^{-k_1}[t_2, d_2]_{k_2-b} + q^{-(k_1+k_2)}[t_3, d_3]_{k_3-b} + \cdots + q^{-(k_1+\cdots+k_{n-1})}[t_n, d_n]_{k_n-b}\Big)\sigma_b.$$

which can be reduced to

$$q^b(1 - q^{-\sum(k_i-b)})P\sigma_b.$$

We chose the k_i so that $\sum k_i = -n\sum a_i$. Thus $-\sum(k_i - b) = n(b + \sum a_i)$. By our assumption on b, either all $a_i = 0$ or $c = q^b(1 - q^{n(b+\sum a_i)})$ is nonzero.

If some $a_i \neq 0$, then $[c^{-1}\tilde{Q}, x] = P$ as required. However, if all $a_i = 0$, then $P = \partial^n$. In this case, $Q = \frac{1}{n+1}\partial^{n+1}$ is the required operator. □

Now we are now ready to prove our main theorem.

Proof of Theorem 2.0.1. Let A^n be the $\Bbbk$-module generated by

$$\{\partial^{\beta^I}\sigma_a \mid \ |I| \leq n \text{ and } a \in \mathbb{Z}\ \}.$$

Since σ_a commutes with ∂^{β^I} up to multiplication by a scalar in $\Bbbk$, we have

$$D_q^0 A^n D_q^0 = RA^nR$$

for any n. We will show by induction on n that $RA^nR = D_q^n$. Since $A^0 = \Bbbk\langle\sigma_a \mid a \in \mathbb{Z}\rangle$, we have $RA^0R = D_q^0$. This proves the base case.

Now suppose that $RA^{n-1}R = D_q^{n-1}$. By Corollary 2.0.1 above, we know that $A^n \subset D_q^n$. Thus, we must show $D_q^n \subseteq RA^nR$. Since $D_q^n = D_q^0 \mathrm{Z}_q'^n D_q^0$, and $RA^nR = D_q^0A^nD_q^0$, it suffices to show $\mathrm{Z}_q'^n \subseteq RA^nR$.

Let $\varphi \in Z_q'^n$. Then $[\varphi, x] \in RA^{n-1}R$ by the induction hypothesis. Thus $[\varphi, x] = \sum r_jP_j\sigma_{b_j}s_j$ where $r_j, s_j \in R$, $P_j = \partial^{\beta^{I_j}}$ with $|I_j| \leq n-1$, and $b_j \in \mathbb{Z}$. By Lemma 2.0.4, we can find $Q_j \in A^n$ such that $[Q_j, x] = P_j\sigma_{b_j}$. Put $Q = \sum r_jQ_js_j$.

Then Q is in RA^nR. Since r_j and s_j commute with x, we have $[Q,x] = \sum r_j P_j \sigma_{b_j} s_j$. Hence $[\varphi - Q, x] = 0$. That is, $\varphi - Q = \eta \in D_q^0 \subset RA^nR$. Hence, $\varphi = Q + \eta$, and so $\varphi \in RA^nR$.

We have shown that $A = \bigcup A^n$ generates D_q over R. Moreover, it is clear from the construction of A^n that A is generated as a $\mathbb{k}$-algebra by $\{\partial^{\beta^a} \mid a \in \mathbb{Z}\}$ and $\{\sigma_a \mid a \in \mathbb{Z}\}$. Furthermore, by Note 2, each ∂^{β^a} is in the $\mathbb{k}$-span of $\{\sigma_a \mid a \in \mathbb{Z}\} \bigcup \{\partial, \partial^\beta\}$. Hence, it is enough to show that each σ_a is generated over R by $\partial^{\beta^{-1}}$, ∂, and ∂^β.

To this end, note that $\sigma_{-1} = \partial^{\beta^{-1}} x - x\partial^{\beta^{-1}}$, $\sigma_0 = 1 = \partial x - x\partial$, and $\sigma_1 = \partial^\beta x - x\partial^\beta$. For any positive integer a, $\sigma_{-a} = (\sigma_{-1})^a$ and $\sigma_a = (\sigma_1)^a$. This proves the theorem. □

Remark 2.0.1. *In remark 3.1.9 of* [LR1]*, Lunts and Rosenberg show that $D_q \supset D_\beta \sigma(\Gamma)$, where D_β denotes the ring of β differential operators. In our case, D_β is the $\mathbb{k}$-algebra generated by x, ∂^β with relations $[\partial^\beta, x]_1 = 1$ (for details, see* Construction of Skew Polynomial Rings*, page 7 of* [GW]*). Our work shows that $D_q \neq D_\beta\sigma(\Gamma)$. Indeed, the usual derivation $\partial \notin D_\beta\sigma(\Gamma)$. By degree considerations, if $\partial \in D_\beta\sigma(\Gamma)$, then $\partial = f\partial^\beta$ where f is of degree 0. Hence, $f = \sum_{-n\le i\le m} c_i\sigma^i$. As $f(x^r) = \frac{r}{(1+q+\cdots+q^{r-1})}x^r$ for $r \ge 1$, we can choose $r >> 0$ such that $\sum_{-n\le i\le m} c_i q^{ri}(1+q+\cdots+q^{r-1}) = r$ implies that $c_{-n}, c_m = 0$.*

Theorem 2.0.2. *The ring D_q is a simple ring.*

Proof. Let $\mathcal{I}$ be an ideal in D_q. Let $0 \neq f \in \mathcal{I}$. Then f can be written as

$$f = \sum_{\{a, I \mid |I| \gneq 0\}} \sigma(a) p_I(x) \partial^{\beta^I} + \sum_{n\in\mathbb{Z}} \sigma(n) p_n(x),$$

where p_I, p_n are polynomials in x, $a \in \mathbb{Z}$ and the multi indices I have entries in $\{0,1\}$. We induct on $d = max\{|I| \mid p_I \neq 0\}$, to claim that the ideal containing f should contain 1. Assume that $d = 0$. Then

$$f = \textit{lower degree in } x + (c_1\sigma(a_1) + \cdots + c_r\sigma(a_r))x^m,$$

where $c_i \in \Bbbk$ and $a_1 \lneq a_2 \lneq \cdots \lneq a_r$. Now $[\sigma(-a_r)f, x]$ have fewer monomials than f. Continuing thus, we can assume that $f = cx^b$ for some $c \in \Bbbk$. Since $[\partial, x^b] = bx^{b-1}$ the base case is proved.

Assume that the claim has been proved for all positive integers less than d. We can write f as

$$f = \textit{lower lengths in } I + \sigma(a_1)p_{I_1}(x)\partial^{\beta^{I_1}} + \cdots + \sigma(a_r)p_{I_r}(x)\partial^{\beta^{I_r}},$$

where $a_s \in \mathbb{Z}, a_1 \lneq a_2 \lneq \cdots \lneq a_r$ and $p_{I_s}(x)$ are polynomials in x where I_s are multi indices of length d, and these multi indices can repeat in the above sum. This implies that,

$$[\sigma(-a_r)f, x] = \textit{lower lengths in } I + \sigma(b_1)g_{I_1}(x)\partial^{\beta^{I_1}} + \cdots + \sigma(b_{r-1})g_{I_{r-1}}(x)\partial^{\beta^{I_{r-1}1}}$$

where $b_s \in \mathbb{Z}, g_{I_l} \in \Bbbk[x]$ and $b_1 \lneq b_2 \lneq \cdots \lneq b_{r-1}$. This shows that $[\sigma(-a_r)f, x]$ has fewer monomials which correspond to multi indices of length d. This completes the theorem. □

3. An intrinsic description of D_q

Although we know that the ring D_q is generated over R by $\partial^{\beta^{-1}}$, ∂, and ∂^{β}, the relations among these are not necessarily apparent. In this section, we will give a description of D_q in terms of generators and relations.

We begin with a study of the grading on D_q. First, to simplify notation, let $\tau = x\partial$ and $\sigma = \sigma_1$. Then $\sigma^{-1} = \sigma_{-1}$.

Lemma 3.0.5. *The ring $(D_q)_0$ of homogeneous q-differential operators of degree 0 is $\Bbbk[\sigma, \tau, \sigma^{-1}]$.*

Proof. It is clear that σ, σ^{-1}, and τ are homogeneous of degree 0. We must show that they generate all of $(D_q)_0$, that they commute, and that they have no other relations among them.

Suppose that $\phi \in (D_q)_0$. Then by Theorem 2.0.1, ϕ can be written as $\phi = \sum P_i$ where each P_i is a monomial in x, $\partial^{\beta^{-1}}$, ∂, and ∂^{β}. Moreover, each P_i has degree 0,

so each P_i has exactly one x for each $\partial^{\beta^{-1}}$, ∂, and ∂^{β}. Since $\partial^{\beta^a} x = q^a x \partial^{\beta^a} + 1$ for $a = -1, 0, 1$, we can rewrite ϕ so that each P_i has the form $P_i = (x\partial^{\beta^{a_1}}) \cdots (x\partial^{\beta^{a_k}})$. If $a_j \neq 0$, then $(\sigma_{a_j} - 1)/(q^{a_j} - 1) = x\partial^{\beta^{a_j}}$. Since $\tau = x\partial^{\beta^0}$, we have that each P_i is in the span of σ, σ^{-1}, and τ. Hence $(D_q)_0$ has the required generators.

Since σ and σ^{-1} are multiplicative inverses, they commute. Comparing the values of $\tau\sigma$ and $\sigma\tau$ on x^m for any m shows that σ and τ also commute. All we need to show now is that there are no remaining relations.

Suppose that we have a relation $\sum a_{ij}\sigma^i\tau^j = 0$ for some $a_{ij} \in \Bbbk$. Then we can multiply this expression by q and σ an appropriate number of times to ensure that all all powers of q in the a_{ij} and all i are nonnegative. Then for every m, $0 = \sum a_{ij}\sigma^i\tau^j(x^m) = (\sum a_{ij} q^{im} m^j) x^m$. Fix an $m >> 1$ such that for every i and j, all the powers of q appearing in a_{ij} are less than all the powers of q appearing in $a_{(i+1)j} q^m$. Then the polynomial $\sum a_{ij} q^{im} m^j$ is a polynomial in q with no terms canceling. Since it is zero, all its coefficients are zero. But if λ is the coefficient of q^k in α_{ij}, then $m^j\lambda$ is the coefficient of q^{im+k} in $\sum a_{ij} q^{im} m^j$. Hence $\lambda = 0$. It follows that $a_{ij} = 0$ completing the proof. □

Corollary 3.0.2. *The ring D_q is a domain.*

Proof. First we show that x cannot be a zero divisor. It is obvious that $x \cdot \varphi \neq 0$ if $\varphi \neq 0$. Suppose $\varphi \cdot x = 0$, then $\varphi(x^n) = 0$ for all positive integers n. Also note that $[\varphi, x]_a = -q^a x\varphi$. Hence φ does not satisfy Lemma 2.0.2.

Suppose $\phi\psi = 0$. Let ϕ_a and ψ_b be the highest degree parts of ϕ and ψ of degrees a and b respectively. Then $\phi_a\psi_b = 0$.

If b is positive, then $(\phi_a x^b)\psi' = 0$ where ψ' is of degree 0. Note also that $\phi_a x^b$ is a non-zero homomorphism. If b is negative, then $\phi_a(\psi_b x^{-b}) = 0$ and $\psi_b x^{-b} \neq 0$ is of degree 0. Hence, we can assume that $b = 0$.

Now, if a is positive, then $x^a\phi'\psi_b = 0$ implies $\phi'\psi_b = 0$, where ϕ' is of degree 0. If a is negative then $(x^{-a}\phi_a)\psi_b = 0$ where $x^{-a}\phi_a$ is of degree 0. Hence, we can assume that $a = 0$ also.

Now the corollary follows from the lemma 3.0.5. □

Now we can describe the ring D_q intrinsically.

Theorem 3.0.3. *The ring D_q of q-differential operators on R is the $\Bbbk$-algebra generated by x, $\partial^{\beta^{-1}}$, ∂^{β^0}, and ∂^{β^1} subject to the relations*

$$\partial^{\beta^a}x - q^a x\partial^{\beta^a} = 1, \quad \partial^{\beta^a}x\partial^{\beta^b} = \partial^{\beta^b}x\partial^{\beta^a}, \quad \textit{and} \quad \partial^{\beta^{-1}} - q\partial^{\beta} = (1-q)\partial^{\beta^{-1}}x\partial^{\beta}.$$

Proof. Let F be the $\Bbbk$-algebra generated by symbols x, $\partial^{\beta^{-1}}$, ∂^{β^0}, and ∂^{β^1} subject to the relations

$$\partial^{\beta^a}x - q^a x\partial^{\beta^a} = 1 \text{ for } \ a = -1, 0, 1. \tag{3.0.5}$$

Giving each ∂^{β^a} degree -1 and giving x degree 1 makes F into a graded $\Bbbk$-algebra. In fact, the natural quotient map $\pi : F \to D_q$ preserves this grading. Let I be the kernel of π. Then I is generated by homogeneous elements.

Suppose that $\theta \in I$ is a homogeneous generator of I of degree n. If $n > 0$, then $\theta\partial^n$ is a homogeneous element of I of degree 0. If $n < 0$, then $x^{-n}\theta$ is likewise a homogeneous element of I of degree 0. Since neither x nor ∂ are zero divisors in F, neither $x^{-n}\theta$ nor $\theta\partial^n$ are zero. Hence every homogeneous generator of I is a factor of some degree 0 element of I.

Now let us restrict our attention to degree 0. The map π takes the ring F_0 of degree 0 elements of F to the ring $(D_q)_0$ with kernel I_0. Using (3.0.5), we can write every element F_0 as a polynomial in $x\partial^{\beta^{-1}}$, $x\partial^{\beta^0}$, and $x\partial^{\beta^1}$. Since $(D_q)_0$ is commutative, the commutators of any two elements of F_0 must be in I_0. Hence,

$$x\partial^{\beta^a}x\partial^{\beta^b} - x\partial^{\beta^b}x\partial^{\beta^a} \in I_0$$

for any a and b.

The only remaining identity in $(D_q)_0$ comes from the fact that σ_1 and σ_{-1} are multiplicative inverses. Since $\sigma_a = 1 + (q^a - 1)x\partial^{\beta^a}$ for any a, we have

$$\begin{aligned} 1 &= \sigma_1\sigma_{-1} \\ &= (1 + (q-1)x\partial^{\beta^1})(1 + (q^{-1}-1)x\partial^{\beta^{-1}}) \\ &= 1 + (q-1)x(\partial^{\beta^1} - q^{-1}\partial^{\beta^{-1}} + (q^{-1}-1)\partial^{\beta^1}x\partial^{\beta^{-1}}) \end{aligned}$$

Hence,

$$(\partial^{\beta^1} - q^{-1}\partial^{\beta^{-1}} + (q^{-1}-1)\partial^{\beta^1} x \partial^{\beta^{-1}}) \in I_0.$$

Let us recapitulate: In $(D_q)_0$, we have only the standard relations

$$\partial^{\beta^a} x - q^a x \partial^{\beta^a} = 1,$$

the commutator relations

$$x\partial^{\beta^a} x \partial^{\beta^b} = x\partial^{\beta^b} x \partial^{\beta^a},$$

and the special relation

$$x(\partial^{\beta^1} - q^{-1}\partial^{\beta^{-1}} = (1-q^{-1})\partial^{\beta^1} x \partial^{\beta^{-1}}).$$

Since all relations in D_q come from elements in I, and all generators of I are factors of elements in I_0, the only other possible relations in D_q are $\partial^{\beta^a} x \partial^{\beta^b} = \partial^{\beta^b} x \partial^{\beta^a}$ and $\partial^{\beta^1} - q^{-1}\partial^{\beta^{-1}} = (1-q^{-1})\partial^{\beta^1} x \partial^{\beta^{-1}}$. Inspecting these last two expressions on x^m shows that they do indeed hold. □

The following formulae are immediate, and hence we do not provide any proofs:

1. $x[\partial, \partial^\beta]_1 = \partial - \partial^\beta$; $[\partial, \partial^\beta]_1 x = \partial - q\partial^\beta$.
2. $(\tau + k)\partial^{\beta^a} = \partial^{\beta^a}(\tau + k - 1)$.
3. $(\tau + 1)\partial^\beta = (\frac{q\sigma - 1}{q-1})\partial$.

   This can be generalized to multi-indices $I = (i_1, i_2, \cdots, i_n)$ (using the notations as in 2.0.2) with entries in $\{0,1\}$ as

$$\left(\prod_{\{j|i_j=1\}} (\tau + j)\right) \partial^{\beta^I} = \left(\prod_{\{j|i_j=1\}} \frac{q^j\sigma - 1}{q-1}\right) \partial^{|I|}$$

Remark 3.0.2. *We have not been successful in determining whether D_q is left noetherian or not.*

4. Generalization to several variables

Let $q_1, q_2, \cdots, q_n$ be transcendental elements over $\mathbb{Q}$ and let $\mathbb{k}$ be a field containing $\mathbb{Q}(q_1, q_2, \cdots, q_n)$. Let $R = \mathbb{k}[x_1, x_2, \cdots, x_n]$, and D_q^m denote $D_q^m(R)$ (respectively D_q denote $D_q(R)$). The ring R has a $\mathbb{Z}^n$-grading given by

$$deg(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}) = (a_1, a_2, \cdots, a_n).$$

Let $\beta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{k}^\times$ be a bicharacter defined by

$$\beta(\mathbf{a}, \mathbf{b}) = q_1^{a_1 b_1} q_2^{a_2 b_2} \cdots q_n^{a_n b_n}.$$

For $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ let $\mathbf{x}^\mathbf{a} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. For each $i, 1 \leq i \leq n$, define $\partial_i^{\beta^k}$ as

$$\partial_i^{\beta^k}(\mathbf{x}^\mathbf{a}) = \frac{(q_i^{k a_i} - 1)}{(q_i - 1)} \mathbf{x}^{\mathbf{a} - (0,0,\cdots,1_i,0,\cdots,0)}.$$

Note that

$$[\partial_i^{\beta^k}, x_j] = 0 \textit{ for } i \neq j,$$
$$[\partial_i^{\beta^k}, \partial_j^{\beta^m}] = 0 \textit{ for } i \neq j,$$
$$[\partial_i^{\beta^k}, \sigma_\mathbf{a}] = 0 \textit{ when } a_i = 0.$$

The notes 2, lemma 2.0.3 and corollary 2.0.1 follow verbatim.

Theorem 4.0.4. *The ring D_q of q-differential operators on R is generated as a $\mathbb{k}$-algebra by R and the set $\{\partial_i^{\beta^{-1}}, \partial_i, \partial_i^{\beta}\}_{1 \leq i \leq n}$.*

Proof. Let A denote the $\mathbb{k}$-algebra generated by R and the set $\{\partial_i^{\beta^{-1}}, \partial_i, \partial_i^{\beta}\}_{1 \leq i \leq n}$. Given $f_i \in R$ and P_i in terms of monomials consisting of $\{\partial_j^{\beta^k} | k, j \in \mathbb{Z}\}$, let Q be such that $[Q, x_i] = P_i \sigma_{\mathbf{a}_i}$. We show that $Q \in A$. We can think of $[., x_i]$ as ∂_{y_i},

where

$$\partial_{y_i}(\partial_j) = \delta_{i,j},$$

$$\partial_{y_i}(\partial_j^{\beta}) = \delta_{i,j}\sigma_{(0,0,\cdots,1_i,0,\cdots 0)},$$

$$\partial_{y_i}(x_j) = 0, \ \ \textit{and}$$

$$\partial_{y_i}(\sigma_{\mathbf{a}}) = (q^{a_i} - 1)x_i\sigma_{\mathbf{a}}.$$

By lemma 2.0.4, for each i, we can find a 'y_i-integral' Q_i of $f_iP_i\sigma_{\mathbf{a}_i}$ in A. Since

$$[[Q, x_i], x_j] = [[Q, x_j], x_i],$$

we have $\partial_{y_i}\partial_{y_j} = \partial_{y_j}\partial_{y_i}$. Thus, we can find an $F \in A$ such that $[Q - F, x_i] = 0$ for all i. That is, $Q - F \in D_q^0$ which is contained in A. Hence the theorem. □

Remark 4.0.3. *The ring D_q is simple and a domain.*

5. Relationship with the Quantum group on sl_2

5.1. **The ring $\Gamma_q(\mathbb{P}^1)$.** Fix $\beta : \mathbb{Z} \times \mathbb{Z} \to \Bbbk^\times$ be fixed as $\beta(n, m) = q^{nm}$. For this β, we define the following rings. Let D_q (respectively L_q) denote the ring of q-differential operators on $\Bbbk[x]$ (respectively $\Bbbk[y]$) with the set-up as in section 2 (respectively, degree of y = -1). L_q is a $\Bbbk$-algebra generated by $\partial_y, \partial_y^{\beta}, \partial_y^{\beta^{-1}}$ where ∂_y^{β} is a left σ_y-derivation for $\sigma_y(y) = \frac{1}{q}y$ and $\partial_y^{\beta^{-1}}$ is a left σ_y^{-1}-derivation. Hence we have the following formulae:

$$\partial_y(y^n) = ny^{n-1},$$

$$\partial_y^{\beta}(y^n) = \left(\frac{1 - \frac{1}{q^n}}{1 - \frac{1}{q}}\right) y^{n-1},$$

$$\partial_y^{\beta^{-1}}(y^n) = \left(\frac{1 - q^n}{1 - q}\right) y^{n-1}.$$

Similarly, let M_q denote the ring of q-differential operators on $\Bbbk[x, x^{-1}]$ where $\Bbbk[x, x^{-1}]$ is $\mathbb{Z}$-graded as $deg(x) = 1$ and $deg(x^{-1}) = -1$. By theorem 3.2.2 of [LR1], there are canonical ring homomorphisms (injective) $D_q \to M_q$ and $L_q \to M_q$.

Specifically, the maps are

$$\begin{array}{cc} D_q \to M_q & L_q \to M_q \\ \sigma^{\pm}(y^n) = q^{\mp n} y^n & \sigma_y^{\pm}(x^n) = q^{\pm n} x^n, \\ \partial(y) = -y^2 & \partial_y(x) = -x^2, \\ \partial^{\beta}(y) = -\frac{1}{q} y^2 & \partial_y^{\beta}(x) = -q x^2, \\ \partial^{\beta^{-1}}(y) = -q y^2 & \partial_y^{\beta^{-1}}(x) = -\frac{1}{q}(x^2), \end{array}$$

and extend the respective derivations (or β-derivations) to the entire ring. We define $\Gamma_q(\mathbb{P}^1)$ as

$$\Gamma_q(\mathbb{P}^1) := Ker \hookrightarrow D_q \bigoplus L_q \to M_q;$$
$$(\varphi_1, \varphi_2) \mapsto \varphi_1 - \varphi_2.$$

For simplicity, let $\Gamma_q := \Gamma_q(\mathbb{P}^1)$. Note that Γ_q is a ring because the homomorphisms $D_q \to M_q$ and $L_q \to M_q$ preserve multiplication.

Lemma 5.1.1. *Γ_q is generated over $\Bbbk$ by the set*

$$\{(\partial, -y^2\partial_y), (-x^2\partial, \partial_y), (\partial^{\beta}, -\frac{1}{q} y^2 \partial_y^{\beta}), (-qx^2\partial^{\beta}, \partial_y^{\beta}),$$
$$(\partial^{\beta^{-1}}, -qx^2\partial_y^{\beta^{-1}}), (-\frac{1}{q}x^2\partial^{\beta^{-1}}, \partial_y^{\beta^{-1}})\}.$$

Proof. Let the algebra generated over $\Bbbk$ by the set mentioned in the statement of the lemma be G.

Clearly, members of the set $\{\partial, \partial^{\beta}, \partial^{\beta^{-1}}, x^2\partial, x^2\partial^{\beta}, x^2\partial^{\beta^{-1}}\}$ map $\Bbbk[x^{-1}]$ to itself. Note that

$$\sigma = \left(\frac{q-1}{q+1}\right)[\partial^{\beta}, x^2\partial^{\beta}]_2 + 1,$$
$$\sigma^{-1} = \left(\frac{\frac{1}{q}-1}{\frac{1}{q}+1}\right)[\partial^{\beta^{-1}}, x^2\partial^{\beta^{-1}}]_{-2} + 1,$$
$$\tau = x\partial = \frac{1}{2}[\partial, x^2\partial].$$

Thus, elements of $(D_q)_0$ (degree 0 endomorphisms in D_q) map $\mathbb{k}[x^{-1}]$ to itself. If $(\varphi_1, \varphi_2) \in \Gamma_q$, then $\varphi_1(\mathbb{k}[y]) \subset \mathbb{k}[y]$ and $\varphi_2(\mathbb{k}[x]) \subset \mathbb{k}[x]$ and $deg(\varphi_1) = deg(\varphi_2)$. If $n = deg(\varphi_1) \leq 0$, then $\varphi_1 = \sum_{|I|=-n} f_I \partial^{\beta^I}$, where $f_I \in (D_q)_0$. Therefore φ_1 is generated by $\{\partial, \partial^\beta, \partial^{\beta^{-1}}\}$. Thus, $(\varphi_1, \varphi_2) \in G$. Similarly, if $n = deg(\varphi_2) \geq 0$, then $\varphi_2 = \sum_{|I|=n} g_I \partial_y^{\beta^I}$ where $g_I \in (D_q(\mathbb{k}[y]))_0$. Again $(\varphi_1, \varphi_2) \in G$. Hence the lemma. $\square$

5.2. **Quantum group on sl_2.** Let U_q denote the Quantum group corresponding to the Lie algebra $sl_2(\mathbb{k})$. That is, U_q is a $\mathbb{k}$-algebra generated by E, F, K, K^{-1}, with relations given by (for details see[CP] or [J])

$$
\begin{aligned}
KK^{-1} = & 1 = K^{-1}K, \\
KEK^{-1} &= q^2 E, \\
KFK^{-1} &= q^{-2} F, \\
EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}
$$

Recall that q is a transcendental element over $\mathbb{Q}$ throughout this paper, and $\mathbb{k}$ contains $\mathbb{Q}(q)$. The ring U_q is a Hopf-algebra and for the purposes of this paper, we will give the comultiplication map Δ:

$$
\begin{aligned}
\Delta(E) &= E \otimes 1 + K \otimes E, \\
\Delta(F) &= 1 \otimes F + F \otimes K^{-1}, \\
\Delta(K) &= K \otimes K.
\end{aligned}
$$

Let the ring corresponding to the *quantum plane* be S. That is,

$$S = \mathbb{k} < u, v > \diagup uv = qvu.$$

There is an action of U_q on this ring given by

$$\begin{aligned} K(1) &= 1 & E(1) &= F(1) = 0, \\ K(u) &= qu & K(v) &= \frac{1}{q}v, \\ E(u) &= 0 & E(v) &= u, \\ F(u) &= v & F(v) &= 0, \end{aligned}$$

and extend the action on S via Δ. If we consider the Ore set $\{v^n\}_{n\geq 0}$ in S, then this action of U_q extends (via Δ) to the Ore-localization. For example, the extension of E to the Ore-localization of S is as follows:

$$\begin{aligned} 0 = E(1) = E(v\frac{1}{v}) &\\ &= E(v)\frac{1}{v} + K(v)E(\frac{1}{v}) \\ &= u\frac{1}{v} + \frac{1}{q}vE(\frac{1}{v}). \end{aligned}$$

Hence, $E(\frac{1}{v}) = -q\frac{1}{v}u\frac{1}{v}$. This extended action of U_q keeps the polynomial ring $\Bbbk[u\frac{1}{v}]$ invariant. We let $x := u\frac{1}{v}$. Then we have a homomorphism of $\Bbbk$-algebras

$$\alpha : U_q \to D_q,$$

given by

$$\begin{aligned} \alpha(F) &= q^{-1}\sigma^{-2}\partial^{\beta^2} = q^{-1}\sigma^{-2}\left(\frac{1+q\sigma}{1+q}\right)\partial^{\beta}, \\ \alpha(E) &= -q^2x^2\partial^{\beta^2} = -q^2\left(\frac{1+q^{-1}\sigma}{1+q}\right)x^2\partial^{\beta}, \\ \alpha(K) &= \sigma^2, \\ \alpha(K^{-1}) &= \sigma^{-2}. \end{aligned}$$

Note that $\alpha(U_q) \subset D_\beta\sigma(\mathbb{Z})$ (referred to in remark 2.0.1) and hence α is not surjective. Similarly, there is an algebra homomorphism

$$\gamma : U_q \to L_q,$$

given by

$$\begin{aligned}\gamma(F) &= -\frac{1}{q^2}\sigma_y^{-2}y^2\partial_y^{\beta^2} = -\frac{1}{q^2}\sigma_y^{-2}\left(\frac{1+q\sigma_y}{1+q}\right)y^2\partial_y^{\beta},\\ \gamma(E) &= q\partial_y^{\beta^2} = q\left(\frac{1+q^{-1}\sigma_y}{1+q}\right)\partial_y^{\beta},\\ \gamma(K) &= \sigma_y^2,\\ \gamma(K^{-1}) &= \sigma_y^{-2}.\end{aligned}$$

Again, γ is not a surjection. The above two homomorphisms give a homomorphism

$$\begin{aligned}\eta : U_q &\to \Gamma_q;\\ u &\mapsto (\alpha(u), \gamma(u)).\end{aligned}$$

Since $(\partial, -y^2\partial_y) \notin \eta(U_q)$, we have the following

Proposition 5.2.1. *η does not give a surjection of U_q to Γ_q.*

But we have a surjection by considering inverse limits. This is shown in the following two subsections.

5.3. **Inverse limits of Γ_q.** Let $\mathcal{A}= \mathbb{Q}[q, q^{-1}]_{(q-1)}$. Let $D_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x])$ denote the ring of $\mathcal{A}/(q-1)^n$ -linear q-differential operators on $\mathcal{A}/(q-1)^n[x]$. Let $D_t(\mathbb{Q}[x,t]/t^n)$ denote the ring of $\mathbb{Q}[t]/t^n$-linear usual differential differential operators on $\mathbb{Q}[x,t]/t^n$. That is, $D_t(\mathbb{Q}[x,t]/t^n) = \mathbb{Q}[t] < x, \partial > \diagup t^n, [\partial, x] = 1$.

Lemma 5.3.1. *The rings $D_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x])$ and $D_t(\mathbb{Q}[x,t]/t^n)$ are isomorphic as $\mathbb{Q}$-algebras.*

Proof. First we note that

$$\begin{aligned}(\mathcal{A}/(q-1)^n)[x] &\cong \mathbb{Q}[x,t]/t^n;\\ (q-1) &\mapsto t,\end{aligned}$$

as $\mathbb{Q}$-algebras. So, it suffices to show that if $d \in D_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x])$ then

$$[\cdots[[d,x],x],\cdots x] = 0$$

for some finite number of commutators. Suppose $\varphi \in D^0_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x])$ is such that $[\varphi, x]_m = 0$ for some m. Then for any $c \in \mathcal{A}/(q-1)^n[x]$, we have $[c\varphi, x] = (q^m - 1)xc\varphi$. Thus, the n-commutators $[\cdots[[c\varphi, x], x]\cdots] = 0$.

Now suppose that $[\varphi, x]_m = d \in D^{l-1}_{q,\mathcal{A}}$. Then, $[c\varphi, x] = (q^m - 1)xc\varphi + cd$. Induction and the fact that $(q-1)^n = 0$ completes the lemma. $\square$

Remark 5.3.1. *In general, we can prove the following ([I] lemma 1.0.0.23): Let $\Bbbk = \mathbb{Q}[t]/t^n$. Let R be a Γ-graded $\Bbbk$-algebra. Suppose $\beta : \Gamma \times \Gamma \to \Bbbk^\times$ be a bicharacter such that $\beta(a, b) - 1 \in (t)$. Futher assume that $\bar{R} := R/t$ is commutative. Then*

$$D_q(R) = \{\varphi \in grHom_{\Bbbk}(R, R) | \textit{there exists an } n \textit{ such that}$$
$$[\cdots[[\varphi, a_1], a_2], \cdots, a_n] = 0, a_i \in R\}$$

Remark 5.3.2. *We have*

$$\lim_{\leftarrow n} D_{q,\mathcal{A}}((A/(q-1)^n)[x]) \cong \mathbb{Q} < x, \partial > [[t]]/[\partial, x] = 1,$$

as $\mathbb{Q}$ -algebras.

Definition.

1. *For each $n \geq 1$, let*

$$\Gamma_{q,n} := Ker \hookrightarrow D_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x]) \bigoplus D_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x^{-1}]$$
$$\to D_{q,\mathcal{A}}(\mathcal{A}/(q-1)^n[x, x^{-1}]);$$
$$(\varphi_1, \varphi_2) \mapsto \varphi_1 - \varphi_2,$$

   the algebra of $\mathcal{A}$-linear global q-differential operators on $\mathbb{P}^1_{n\mathcal{A}}$.
2. *The $\Gamma_{q,n}$ form an inverse system with $\hat{\Gamma}$ as its inverse limit.*

Remark 5.3.3.

1. *If $(\varphi_1, \varphi_2) \in \Gamma_{q,n}$ then $\varphi_1 \in (\mathcal{A}/(q-1)^n) < \partial_x, x^2\partial_x >$ and $\varphi_2 \in (\mathcal{A}/(q-1)^n) < \partial_{x^{-1}}, x^{-2}\partial_{x^{-1}} >$.*

2. *The map* $\varphi \in (\mathcal{A}/(q-1)^n) < \partial_x, x^2\partial_x >$ *if and only if* φ *can be written as*

$$\varphi = \sum_{i\geq 1} f_i(q, q^{-1}) g_i(x) \partial_x^i$$

where $g_i(z)\partial_x^i \in \mathbb{Q} < \partial_x, x^2\partial_x >$. *Moreover, if image of* φ *is contained in* $(q-1)^{n-1}\mathcal{A}/((q-1)^{n-1}[x]$, *then by induction on* n, *we can see that* $f_i = (q-1)^{n-1}h_i$. *Thus, there exists* $d \in \mathcal{A}/(q-1)^n < \partial_x, x^2\partial_x >$ *such that* $\varphi = (q-1)^n d$.

5.4. **Inverse limit of U_q.** Let $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[m]! = \prod_{1\leq i\leq m}[m]$. Denote by

$$E^{(m)} = \frac{E^m}{[m]!}, F^{(m)} = \frac{F^m}{[m]!}, m \in \mathbb{Z}.$$

Let $U_{q,\mathcal{A}}$ be the subalgebra of U_q generated by $E^{(m)}, F^{(m)}, K, K^{-1}, m \in \mathbb{Z}$ over $\mathcal{A}$. For each $n \geq 1$, let $U_{q,n}$ denote the ring $U_{q,\mathcal{A}}/(q-1)^n$, and $\hat{U}_q$ denote their inverse limit. There are homomorphisms α_n, γ_n induced by α, γ (defined in the subsection 5.2) respectively, giving a map $\eta_n : U_{q,n} \to \Gamma_{q,n}$, whose inverse limit is denoted by $\hat{\eta} : \hat{U}_q \to \hat{\Gamma}$.

Theorem 5.4.1. *The map* $\hat{\eta} : \hat{U}_q \to \hat{\Gamma}$ *is a surjection.*

Proof. We show by induction that η_n is surjective for $n \geq 1$. When $n = 1$, note that $\Gamma_{q,1}$ is generated over $\mathbb{Q}$ by $(\partial, -x^{-2}\partial_{x^{-1}}), (-x^2\partial, \partial_{x^{-1}})$. The map η_1 is clearly surjective. Let $(\varphi_1, \varphi_2) \in \Gamma_{q,n}$. Consider $(\bar{\varphi_1}, \bar{\varphi_2}) \in \Gamma_{q,n-1}$. Since η_{n-1} is surjective, there exists a $u \in U_{q,\mathcal{A}}$ such that $\eta_{n-1}(\bar{u}) = (\bar{\varphi_1}, \bar{\varphi_2})$. Consider $\eta_n(\bar{u}) - (\varphi_1, \varphi_2) = (\psi_1, \psi_2) \in \Gamma_{q,n}$. Since $(\bar{\psi_1}, \bar{\psi_2}) = 0 \in \Gamma_{q,n-1}$, we have $(\psi_1, \psi_2) = (q-1)^{n-1}(d_1, d_2)$ for $(d_1, d_2) \in \Gamma_{q,n}$ by remark 5.3.3. Consider $(\bar{d_1}, \bar{d_2}) \in \Gamma_{q,n-1}$. Again by surjectivity of η_{n-1}, there exists a $v \in U_{q,\mathcal{A}}$ such that $\eta_{n-1}(\bar{v}) = (\bar{d_1}, \bar{d_2})$. Since $n \gneq 1$, we have $\eta_n((q-1)^n\bar{v}) = (d_1, d_2)$. Thus, $\eta_n(\bar{u} - (q-1)^n\bar{v}) = (\varphi_1, \varphi_2)$. $\square$

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